Junior Mathematical Olympiad

MATHLETE TRAINING CENTRE
PERSEVERENCE RIGOR DEDICATION 224 BISHAN STREET 23 B1-131
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Solutions

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Section A

> **A1.** What is the sum of the first nine primes?

SOLUTION **100**

The first 9 primes are 2, 3, 5, 7, 11, 13, 17, 19, and 23.

> **A2.** Forty-two cubes of side-length 1 cm are stuck together to form a solid cuboid. The perimeter of the base of the cuboid is 16 cm. What is its height, in cm?

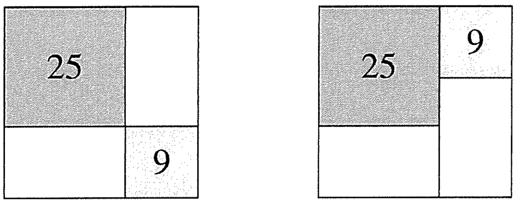
SOLUTION **6 cm**

We need to factorise 42 as $L \times W \times H$, with $2L + 2W = 16$. The only solutions are $L = 7, W = 1, H = 6$ and $L = 1, W = 7, H = 6$.

> **A3.** $PQRS$ is a square which has been divided into four regions: two identical rectangles, one square of area $9\,\text{cm}^2$ and a second square of area $25\,\text{cm}^2$. What is the area of square $PQRS$, in cm^2?

SOLUTION **64** cm^2

There are only two ways to divide up a square into two unequal squares and two identical rectangles:



In each case, the side length of $PQRS$ must be the sum of the side lengths of the smaller squares, in this case 3+5=8, for an area of $64\,\text{cm}^2$.

> **A4.** Note that $49 = 4 \times 9 + 4 + 9$. How many two-digit numbers are equal to the product of their digits plus the sum of their digits?

SOLUTION **9**

A two-digit number $\overline{AB}$ has the value $10A + B$, so we need $10A + B = AB + A + B$. Subtracting B from each side, and dividing by A (since A must be non-zero, to be a two-digit number), we obtain $B = 9$. This condition turns out to be sufficient too (check for yourself that, assuming $B = 9$, we can be sure that the condition will definitely hold), so we get 9 solutions, corresponding to A being one of 1,...,9.

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A5. The difference between an interior angle of a regular polygon and an exterior angle of the same polygon is 150°. How many sides does the polygon have?

Solution **24**

Suppose the shape has n sides. Then the exterior angle is $360°/n$ and the interior angle is $180° - 360°/n$. This tells us that $180 - 720/n = 150$, ie $720/n = 30$, or $n = 24$.

A6. When a group of five friends met up, Alice shook hands with one person; Bill shook hands with two people; Cara shook hands with three people; Dhriti shook hands with four people. How many people did Erin shake hands with?

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Solution **2**

D shook hands with everyone, notably including A. Therefore A didn't shake hands with anyone other than D. C shook hands with all but one, ie all but A, notably including B. B shook hands with only two: C and D. Therefore C and D shook hands with E, while A and B did not, so the answer is 2.

A7. The time is 20:24 (expressed in 24-hour time). What is the angle between the hour hand and the minute hand on an accurate analogue clock, in degrees?

Solution **108°**

In this solution, all positions of hands will be expressed as clockwise angles relative to the 12:00 position at the top of the clock. Note also that, despite the time being expressed in 24-hour time, the analogue clock is of course a 12-hour clock as usual.

At 20:00, the hour hand is at a position of $8/12 * 360° = 240°$ and the minute hand is at 0°. At 20:24, the minute hand is at a position of $24/60 * 360° = 144°$, and the hour hand has moved $1/12$ as far as the minute hand has, ie 12° bringing it to a position of 252°, for a difference between them of 108°. Since this result is less than 180° it is the angle between the two hands.

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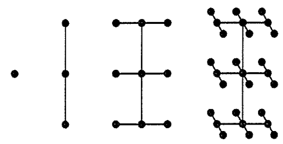
A8. We make a sequence of diagrams. The first diagram consists of a single node. The second diagram is made from the first diagram by drawing two edges of length 1 cm from that node, and putting a node at the other end of each new edge. After that, we make each new diagram from the previous diagram by adding two new edges to each node, with these new edges each having half the length of the edges that were added in the previous diagram. We also attach a node to the end of each new edge. The first four diagrams are shown. Find the total length of all the edges in the fifth diagram, in cm.



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SOLUTION **16.25 cm**

The number of nodes increases by a factor of 3 at each stage. Therefore the number of new edges also increases by a factor of 3 relative to the number of new edges in the previous diagram. Since each new edge is half as long as the new edges in the previous diagram, this means that the total new length is $\frac{3}{2}$ as much as the new length in the previous diagram. Adding these terms up gives $2 + 3 + 4.5 + 6.75 = 16.25$ cm.

A9. The numbers x and y satisfy the equations:

$$xy = \frac{7}{6} \qquad x(y+1) = \frac{5}{3} \qquad y(x+1) = \frac{7}{2}$$

What is the value of $(x+1)(y+1)$?

SOLUTION **5**

There are many ways to solve this question, including explicitly solving for x and y. But an easy way is to multiply the second and third equations together to get $xy(x+1)(y+1) = 35/6$, and then divide that by the first equation to get $(x+1)(y+1) = 5$. Note that this division is safe, as $xy \neq 0$: we are told that it is $\frac{7}{6}$.

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A10. What is the last digit of $2^{(2^{2024})}$?

SOLUTION **6**

"Last digit" questions are quite common, and it's tempting on a question like this to try to work out the last digit of 2^{2024} as a first step. Unfortunately, knowing the last digit of that expression isn't quite enough to answer the question. Instead, consider the last digit of 2^n as we increase n, starting from $n = 1$ (we'll want to work this out with $n = 2^{2024}$). It goes through a regular pattern: 2, 4, 8, 6, 2, 4, 8, 6,... (can you prove this?). Since 2^{2024} is clearly a multiple of 4, the final answer will be the value in the sequence at positions which are a multiple of 4, ie 6. Note that asking for the last digit of something is really asking for the remainder when we divide by 10; here, our first step is to find the remainder when we divide 2^{2024} by 4, which is why finding the last digit of 2^{2024} doesn't directly help us.

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Section B

B1. What is the smallest positive integer that only contains the digits 0 and 1, and is divisible by 36?

SOLUTION

Since $36 = 4 \times 9$, and since 4 and 9 have no factors in common, a number is a multiple of 36 if and only if it is a multiple of both 4 and 9. To be a multiple of 9, the digits must add up to a multiple of 9, which means that the number of 1s must be a multiple of 9. To be a multiple of 4, the last two digits must be a multiple of 4. Since none of 1, 10, nor 11 are multiples of 4, the only option is to put 0s in the last two digits. Since we need to use at least 9 1s and 2 0s, we need at least 11 digits. The solution 11111111100 is the only such 11 digit number, as we need the two 0s to go at the end. Any other solution would have more digits, and would therefore be larger.

B2. Natasha and Rosie are running at constant speeds in opposite directions around a running track. Natasha takes 70 seconds to complete each lap of the track and meets Rosie every 42 seconds.

How long does it take Rosie to complete each lap?

SOLUTION

Let t be the length of the track, in m, and n and r be Natasha's and Rosie's running speeds, respectively, in m/s. From one meeting to the next, they need to have run a total distance of t between the two of them. So we have

$$\begin{aligned} t &= 70n \\ t &= 42(n+r) \\ \implies 42r &= 28n \end{aligned}$$

Rosie's lap time is $\frac{t}{r} = \frac{t}{n}\frac{n}{r} = 70 \times \frac{42}{28} = 105\text{s}$.

An alternative method is to divide up the track into 210 equal sections. It's tempting to assume that it's 210m long, but that's making an assumption beyond what the question tells us so we can't call these sections metres, instead let's call them squigs. Natasha's running speed is 3 squigs/second. The combined running speed of the two of them is 5 squigs/second. Therefore Rosie's speed is 2 squigs/second, so she takes $\frac{210}{2} = 105$ seconds to run a lap. Note that the size of a squig turned out not to matter – but that doesn't mean that we could assume that without justification.

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B3. The positive integers from 1 to n ($n \geq 2$) inclusive are to be spaced equally around the
circumference of a circle so that:

(a) no two even numbers are adjacent;

(b) no two odd numbers are adjacent;

(c) no two numbers differing by 1 are adjacent.

What is the smallest value of n for which the above is possible?

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SOLUTION

$n = 2$ is clearly impossible as 1 and 2 would be next to each other. Therefore the digit 3 must appear. Since the neighbours of 3 must be even, and can't be 2 or 4, the smallest they can be is 6 and 8. This shows that we need $n \geq 8$. All that remains is to check that $n = 8$ is possible. The sequence, going around the circle, could be 1, 6, 3, 8, 5, 2, 7, 4, before coming back to the original 1, which satisfies all the conditions.

B4. My piggy bank contains x pound coins and y pennies and rattles nicely. If instead it contained y pound coins and x pennies, then I would only have half as much money.

What is the smallest amount of money my piggy bank could contain?

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SOLUTION

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The wording here is a bit curious, with the reference to rattling. While it is easy to dismiss it as whimsy, the important conclusion from the rattling is that at least one of x and y is non-zero. The next sentence tells us that

$$100x + y = 2(100y + x)$$

ie $98x = 199y$. The solution $x = y = 0$ is contradicted by the rattling, so we need to find the smallest positive solution. Since the RHS is a multiple of 199, so must the LHS be. Since 98 has no prime factors in common with 199 (199 is, in fact, prime) we know that x must be a multiple of 199. So the smallest solution is $x = 199$, $y = 98$, for a total amount of money of £199.98.

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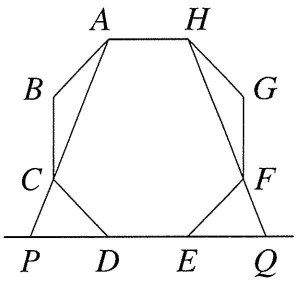
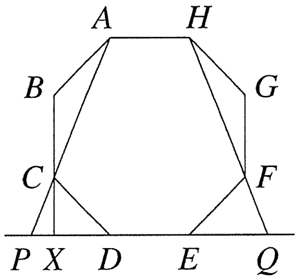
B5. A regular octagon $ABCDEFGH$ has sides of length 1. The lines AC and HF meet the line going through D and E at P and Q respectively.

What is the length of the line PQ?



Solution

Extend line BC until it meets line PQ at point X.



The exterior angles of a regular octagon are 45° so the interior angles are 135°. So $\angle ABC = 135°$.

Triangle ABC is isosceles, as two of its sides are sides of the regular octagon. Therefore we have $\angle BAC = \angle BCA$. Since angles in a triangle add to 180°, these two angles add to 45° and so we have $\angle BAC = \angle BCA = 22.5°$. Since $\angle BCA = \angle PCX$ (vertically opposite), we have $\angle PCX = 22.5°$.

$\angle XCD = 45°$ (exterior angle of the octagon), so $\angle PCD = \angle PCX + \angle XCD = 67.5°$. $\angle CDP = 45°$ (exterior angle) and angles in triangle PCD add to 180° so $\angle CPD = 67.5°$. Thus triangle CPD is in fact isosceles, and so $PD = CD = 1$.

The same argument holds on the other side, giving $PQ = 3$.

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B6. Let A be the set of $2n$ positive integers $1, 2, 3, \ldots, 2n$, where $n \geq 1$.

For which values of n can this be split into n pairs of integers in such a way that every pair has a sum which is a multiple of 3?

As always in Olympiad problems such as this, you also need to explain why no other values of n are possible.

SOLUTION

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Suppose that n is 1 more than a multiple of three. Then $2n$ is one less than a multiple of three, and we have an easy solution: pair the first with the last, the second with the second last, etc. So in this case we can definitely achieve the required splitting.

Suppose n is a multiple of 3, so $n = 3k$. Then the last four integers are $6k - 3, 6k - 2, 6k - 1, 6k$. We can pair these up as $(6k - 3, 6k)$ and $(6k - 2, 6k - 1)$. This leaves a smaller problem, with $n' = n - 2$ taking the role of n in the original problem. Since n' is one more than a multiple of three, these remaining integers can also be split in the required way.

Now suppose that n is 1 less than a multiple of 3. Consider the sum $1 + 2 + \cdots + 2n$. Since $n + 1$ is a multiple of 3, we can split the set $\{1, 2, \ldots, 2(n + 1)\}$ into pairs whose sum is a multiple of 3, as shown above. So $1 + 2 + \cdots + 2(n + 1)$ is a multiple of 3. But $1 + 2 + \cdots + 2(n + 1) = (1 + 2 + \cdots + 2n) + 4n + 3$, and $4n + 3$ is not divisible by 3 as n is not divisible by 3. So $1 + 2 + \cdots + 2n$ is not divisible by 3, and it is impossible to achieve the required split.

Alternatively, starting from the same sum $1 + 2 + \cdots + 2n$, we can pair the terms up from the outside in: $1 + 2n$, $2 + (2n - 1)$, etc, each of which sum to $2n + 1$. There are n such pairs, so the total sum is $n(2n + 1)$. Since neither n nor $2n + 1$ is a multiple of 3 in this case, the product is also not a multiple of 3 (note that this argument relies on 3 being prime).

So in summary, any value of n which is either a multiple of 3, or 1 more than a multiple of 3, can be split in the required way, and no others.

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