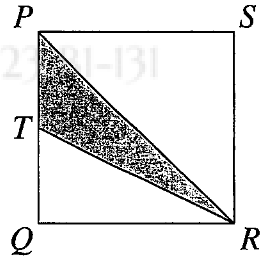
Intermediate Mathematical Challenge

Solutions 2024

1. D The value of $\frac{20+24}{20-24} = \frac{44}{-4} = -11$.

2. A The smallest two-digit prime is 11 and the largest two-digit prime is 97. The difference between them is $97 - 11 = 86$.

3. A As T is the midpoint of the side PQ, $PT = TQ$. Therefore triangles PRT and TRQ have equal bases and the same perpendicular height, namely QR. So they are equal in area. Hence the shaded area is half the area of triangle PQR, which in turn is half the area of the square $PQRS$.



Therefore the required fraction is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

4. D Let the length of the hypotenuse of the triangle be l.
Then, by Pythagoras' Theorem, $l^2 = (\sqrt{5})^2 + (\sqrt{12})^2 = 5 + 12 = 17$.
So the length of the hypotenuse is $\sqrt{17}$.

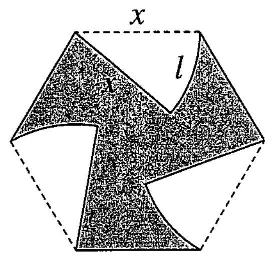
5. C The ages of the youngest three grandchildren are three consecutive positive integers which have a mean of 6. So they are 5, 6, 7. Hence the ages of the other four grandchildren are 8, 9, 10, 11.
Therefore the mean age of the oldest three grandchildren is $\frac{9+10+11}{3} = \frac{30}{3} = 10$.

6. C The distance, d, of the point (x, y) from the origin O, $(0, 0)$, is given by $d = \sqrt{x^2 + y^2}$.
It is clear that the points $(5, 0)$, $(4, 3)$, $(3, 4)$, $(0, 5)$ are all at a distance of 5 units from O as $\sqrt{4^2 + 3^2} = \sqrt{3^2 + 4^2} = 5$. So those four points all lie on a circle with centre O and radius 5.
However, the distance from the point $(2, 2)$ to O is $\sqrt{2^2 + 2^2} = \sqrt{8} \neq 5$.
Therefore the point $(2, 2)$ does not lie on the same circle as the other four points.
(*There is a unique circle through any three non-collinear points, so there can be no circle that passes through the point (2, 2) and three of the other four given points.*)

7. B The approximate number of grains of rice in a can of Penny's rice pudding is $50\,000\,000 \div 25\,000 = 50\,000 \div 25 = 2000$.

8. **C** $999 \times 999 + 999 = 999(999 + 1) = 999 \times 1000 = 999\,000$.

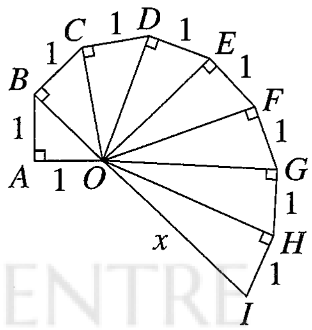
9. **D** Let the length of the side of the hexagon be x mm and the length of the arc of each sector be l mm. Then the radius of each sector is also x mm. Therefore $2x + l = 18$.
The perimeter, in mm, of the shape formed when the three sectors are removed from the hexagon is $6x + 3l = 3(2x + l) = 3 \times 18 = 54$.



10. **B** $\frac{20}{24} + \frac{20}{25} = \frac{5}{6} + \frac{4}{5} = \frac{25 + 24}{30} = \frac{49}{30}$.

11. **C** The horizontal part of the frame consists of two rectangles, each measuring 80 cm by 4 cm. The vertical part of the frame consists of two rectangles, each measuring 72 cm by 4 cm. Therefore the percentage of the area of the square covered by the frame is
$\frac{2(80 \times 4 + 72 \times 4)}{80 \times 80} \times 100\% = \frac{8(80 + 72)}{8 \times 8}\% = \frac{80 + 72}{8}\% = (10 + 9)\% = 19\%$.

12. **D** By Pythagoras' Theorem, $OB^2 = OA^2 + AB^2 = 1^2 + 1^2 = 1 + 1 = 2$.
Similarly, $OC^2 = OB^2 + BC^2 = 2 + 1 = 3$ and $OD^2 = OC^2 + CD^2 = 3 + 1 = 4$. Continuing in the same way, we see that $OE^2 = 5$, $OF^2 = 6, OG^2 = 7, OH^2 = 8, OI^2 = 9$.
So the length of the line segment marked x is $\sqrt{9} = 3$.

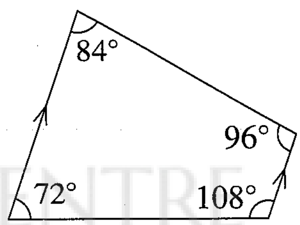


13. **C** Let the certain number be x. Then $\frac{6x}{5} = 2 \times \frac{4}{5}(x - 20)$.

Therefore $6x = 8(x - 20) = 8x - 160$. Hence $2x = 160$, so $x = 80$.

14. **B** The interior angles of the quadrilateral are in order:
$\frac{6}{30} \times 360°, \frac{7}{30} \times 360°, \frac{8}{30} \times 360°, \frac{9}{30} \times 360°$, that is 72°, 84°, 96°, 108°.

Therefore the quadrilateral has two pairs of adjacent angles which sum to 180° (72° + 108° and 84° + 96°), so it is a trapezium, as shown.



To confirm that none of the other options are correct, we note that the quadrilateral does not have a right angle and is not cyclic as opposite angles do not sum to 180°. Furthermore, it is not a kite nor a parallelogram as the former has one pair of equal angles and the latter has two pairs of equal angles.

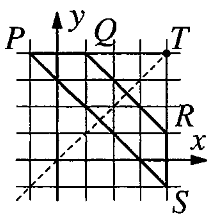
15. **B** Let the weights, in grams, of Carrie, Barrie and Rollie be c, b, r respectively.
Then: $c + b = 4000 + r$; $b + r = c - 2000$; $c + r = 3000 + b$.
Adding these three equations gives: $2c + 2b + 2r = 5000 + r + c + b$. So $c + b + r = 5000$.
Hence $c + b = 5000 - r$.
Substituting in the first of the above equations: $5000 - r = 4000 + r$. Therefore $2r = 1000$.
Hence the weight of Rollie the rat is 500 g.

16. **D** For integer values of n greater than 4, $n!$ is a multiple of 5. So $5! + 6! + 7! + 8! + 9! + 10!$ is divisible by 5.

Therefore the remainder when $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is divided by 5 is the same as when $1!+2!+3!+4!$ is divided by 5. Now $1!+2!+3!+4! = 1+2+6+24 = 33 = 6\times5+3$. Therefore the required remainder is 3.

17. **E** $4^{(3^2)} \div (4^3)^2 = 4^9 \div 4^6 = 4^3 = 64$.

18. **E** The diagram shows that Q is the point (1, 4), R is the point (4, 1) and S is the point (4, −1). Let T be the point (4, 4).
Then the area of triangle PST is $\frac{1}{2} \times 5 \times 5$ and the area of triangle QRT is $\frac{1}{2} \times 3 \times 3$. So the area of quadrilateral $PQRS = \frac{1}{2} \times (25 - 9) = \frac{1}{2} \times 16 = 8$.



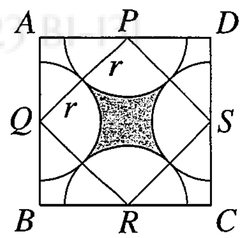
19. **A** Let p, q, r, s, t be missing numbers in the grid, as shown.
Then, comparing the third column with the diagonal running from top right to bottom left: $r \times t \times 3 = r \times 6 \times 2$. So $t = 4$, as r is non-zero.
Comparing the first column with the diagonal running from top left to bottom right: $p \times s \times 2 = p \times 6 \times 3$. So $s = 9$, as p is non-zero.

p	q	r
s	6	t
2	x	3

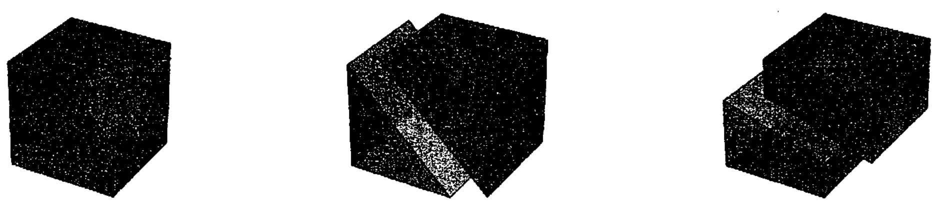
It can now be deduced that the 'magic' product is $9 \times 6 \times 4$. Therefore, considering the bottom row of the grid: $2 \times x \times 3 = 9 \times 6 \times 4$. Hence $x = 9 \times 4 = 36$.
(*It is left as exercise for the reader to complete the square.*)

20. **D** As buying three T-shirts for the price of two is equivalent to a saving of £5.50 on each T-shirt, the money saved using this offer is 3 × £5.50 = £16.50. The money saved is the cost of one T-shirt so this cost is £16.50.
So, using the offer, the cost of three T-shirts is 2 × £16.50 = £33.00.

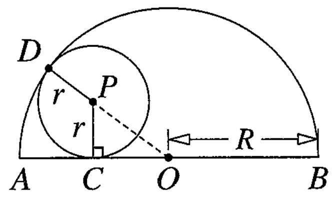
21. **E** Let A, B, C, D, P, Q, R, S be the the points shown and let the radius of each semicircle be r.
Note that PQ passes through the touching point of the two semicircles with centres P and Q. Now consider triangle PAQ. As P and Q are the midpoints of AD and AB respectively, $AP = AQ = 2$. Therefore PAQ is an isosceles right-angled triangle and $\angle APQ = 45°$. Similarly, $\angle DPS = 45°$. Hence $\angle QPS = (180 - 2 \times 45)° = 90°$. So from the symmetry of the figure we see that $PQRS$ is a square which contains the shaded area and four quarter-circles of radius r.
By Pythagoras' Theorem, $PQ^2 = PA^2 + AQ^2$. Therefore $(2r)^2 = 2^2 + 2^2$. Hence $4r^2 = 8$, that is $r^2 = 2$.
So the shaded area is $(2r)^2 - 4 \times \frac{1}{4} \times \pi \times r^2 = 4r^2 - \pi r^2 = 8 - 2\pi$.



22. **E** The cube has six faces. When a single plane cut is made, it splits the cube into two polyhedra. Each of these will have all or part of some of the faces of the cube together with one new face. So it is not possible to produce a polyhedron with 8 faces. The first diagram below shows a cut which gives a polyhedron with 4 faces and a polyhedron with 7 faces. The second diagram shows a cut which gives two polyhedra, each with 5 faces. Finally, the third diagram shows a cut which gives two polyhedra, each with 6 faces.



23. B Let the radii of the semicircle and circle be R and r respectively; let the perpendicular from P to AO meet AO at C. So C is the point where AB touches the circle. Also let the semicircle and the circle touch at the point D.



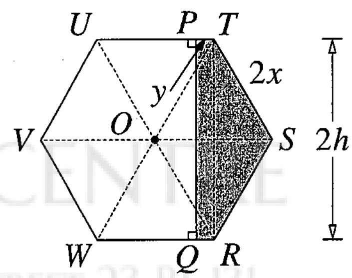
The radii OD and PD are both perpendicular to the common tangent at D, so OPD is a straight line. Hence OP has length $R - r$.

Consider triangles ACP and OCP: $PA = PO$ (given); $\angle ACP = \angle OCP = 90°$; PC is common to both triangles. So the triangles are congruent (RHS) and we deduce that $AC = OC$.
Hence $OC = \dfrac{OA}{2} = \dfrac{R}{2}$.
Applying Pythagoras' Theorem to triangle OCP:
$OP^2 = OC^2 + CP^2$. Therefore $(R - r)^2 = \left(\dfrac{R}{2}\right)^2 + r^2$.

So $R^2 - 2Rr + r^2 = \dfrac{R^2}{4} + r^2$.

Hence $\dfrac{3R^2}{4} = 2Rr$, that is $r = \dfrac{3R}{8}$, as $R \neq 0$. Therefore the required ratio is 3 : 8.

24. C Let the side of the hexagon be of length $2x$. Also, let PT and TR have length y and $2h$ respectively. Let O be the centre of the hexagon.
The area of the hexagon may be divided into six equilateral triangles of side-length $2x$. Also, the height of each triangle is h.
Then the area of the hexagon is $6 \times \frac{1}{2} \times 2x \times h = 6xh$.
The shaded area is the sum of the area of rectangle $PQRT$ and the area of triangle RST. Now the area of triangle RST is equal to the area of triangle OST, so the shaded area is $y \times 2h + \frac{1}{2} \times 2x \times h = 2hy + xh$.



Therefore $2hy + xh = \frac{1}{4} \times (6xh)$, that is $8hy + 4xh = 6xh$.
So $8y = 2x$, as $h \neq 0$. Hence $y = \frac{x}{4}$. Now as Jay walks clockwise around the figure, she walks a distance $2y + 4x$, that is $2 \times \frac{x}{4} + 4x$. So Jay walks $\frac{9x}{2}$ between P and Q.
Therefore the distance which Kay walks between P and Q is $6 \times 2x - \frac{9x}{2} = \frac{15x}{2}$.
Hence the required ratio is $\frac{9x}{2} : \frac{15x}{2} = 9 : 15 = 3 : 5$.

25. E Let the costs of a gold coin and a silver coin be £g and £s respectively.
Then $g = \left(1 + \dfrac{x}{100}\right)s$ and $s = \left(1 - \dfrac{y}{100}\right)g$.
Hence $g/s = (100 + x)/100$ and $s/g = (100 - y)/100$. Therefore $(100 + x)(100 - y) = 10\,000$.

So $x = \dfrac{10\,000}{100 - y} - 100$. Therefore $100 - y$ is a factor of 10 000 and we need consider only values of $100 - y$ less than 100 as y is a positive integer.
Hence the possible values of $100 - y$ are 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80. The corresponding values of x are 9900, 4900, 2400, 1900, 1150, 900, 525, 400, 300, 150, 100, 25.
So there are twelve possible values of x.